

D 110208

(Pages : 3)

Name.....

Reg. No.....

**FIFTH SEMESTER (CBCSS—UG) DEGREE EXAMINATION
NOVEMBER 2024**

Mathematics

MTS 5B 05—ABSTRACT ALGEBRA

(2020 Admission onwards)

Time : Two Hours and a Half

Maximum : 80 Marks

Section A*Answer any number of questions.**Each question carries 2 marks.**Ceiling is 25.*

- Let n be a positive integer. Prove that the congruence class $[a]_n$ has a multiplicative inverse in \mathbb{Z}_n if and only if $(a, n) = 1$.
- Make multiplication table for \mathbb{Z}_6 .
- Find the order of the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{pmatrix}$.
- Let G be a nonempty set with an associative binary operation in which the equations $ax = b$ and $xa = b$ have solutions for all $a, b \in G$. Prove that G is a group.
- Let G be group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.
- Prove that any group of prime order is cyclic.
- In $GL_2(\mathbb{R})$, find the order of $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.
- Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.
- Let G be a cyclic group. If G is infinite, prove that $G \cong \mathbb{Z}$.

Turn over

10. Prove that the set of all even permutations of S_n is a subgroup of S_n .
11. Let $\phi: G_1 \rightarrow G_2$ be group homomorphism, with $K = \ker(\phi)$. Prove that K is a subgroup of G_1 such that $gKg^{-1} \in K$ for all $k \in K$ and $g \in G_1$.
12. Let $G = \mathbb{Z}_{12}$ and $H = \langle 4 \rangle$. Find all cosets of H .
13. State First isomorphism theorem.
14. Let G be a group. Prove that $\text{Aut}(G)$ is a group under composition of functions.
15. Prove that any subring of a field is an integral domain.

Section B

Answer any number of questions.

Each question carries 5 marks.

Ceiling is 35.

16. State and prove Euler theorem.
17. On \mathbb{R}^2 , define $(a_1, a_2) \sim (b_1, b_2)$ if $a_1^2 + a_2^2 = b_1^2 + b_2^2$. Check that this defines an equivalence relation. What are the equivalence classes?
18. Prove that the units of \mathbb{Z}_8 forms a group under multiplication of congruences.
19. Let G be a group with identity element e , and let H be a subset of G . Prove that H is a subgroup of G if and only if the following conditions hold :
 - (a) $ab \in H$ for all $a, b \in H$;
 - (b) $e \in H$; and
 - (iii) $a^{-1} \in H$ for all $a \in H$.
20. Let G be a group, and let H and K be subgroups of G . If $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, Prove that HK is a subgroup of G .
21. Prove that every subgroup of a cyclic group is cyclic.
22. State and prove fundamental theorem of homomorphism.

23. Let G be a group with normal subgroups H, K such that $HK = G$ and $H \cap K = \{e\}$. Prove that $G \cong H \times K$.

Section C

*Answer any two questions.
Each question carries 10 marks.
Maximum 20 marks.*

24. a) Prove that every permutation in S_n can be written as a product of disjoint cycles
- b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 7 & 6 & 3 & 8 & 1 & 4 \end{pmatrix}$ be a permutation in S_8 . Express σ as a product of disjoint cycles.
25. a) Let $\phi: G_1 \rightarrow G_2$ be an isomorphism of groups. Prove that ϕ preserves following structural properties:
- (i) If a has order n in G_1 , then $\phi(a)$ has order n in G_2 ,
 - (ii) If G_1 is abelian, then so is G_2 ,
 - (iii) If G_1 is cyclic, then so is G_2 .
- b) Prove that $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
26. Let H be a subgroup of the group G . Prove that the following conditions are equivalent.
- a) H is a normal subgroup of G ;
 - b) $aH = Ha$ for all $a \in G$;
 - c) for all $a, b \in G$, $ab \in H$ is the set theoretic product $(aH)(bH)$;
 - d) for all $a, b \in G$, $ab^{-1} \in H$ if and only if $a^{-1}b \in H$.
27. State and prove second isomorphism theorem.

(2 × 10 = 20 marks)