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D 50196 **(Pages : 3)** **Name..**

Reg. No....................................

FIFTH SEMESTER (CUCBCSS—UG) DEGREE EXAMINATION NOVEMBER 2023

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

(2018 Admissions)

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. Define a group homomorphism.
2. Write down all the subgroups of the Klein group.
3. Define subring of a ring.
4. Compute $\sigma^{-1} \in S_3$ if $\sigma = (1\ 2)(1\ 2\ 3)^2$.
5. Find all the orbits of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 6 & 5 \end{pmatrix} \in S_6$.
6. Draw the subgroup diagram for $\mathbb{Z}_{11}$.
7. What is the field of quotients of integers ?
8. Define characteristic of a ring.
9. Distinguish between a binary operation and a permutation.
10. If $|a| = 10$, what is $\left|a^{-1}\right|$?
11. Give an example of two distinct but isomorphic rings.
12. Give three distinct proper non-trivial normal subgroups N_1, N_2 and N_3 of $4\mathbb{Z}$ such that $N_1 \leq N_2 \leq N_3$.

(12 × 1 = 12 marks)

Turn over

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Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 marks.

13. Give an example or counter example for the statement that union of two subgroups is a subgroup.
14. When is the kernel of a homomorphism a trivial subgroup ? Establish your claim.
15. Are the fields Z_{11} and Z_{13} isomorphic ? Give justification(s) to your answer.
16. State Legrange's theorem and prove an important corollary of this theorem.
17. Solve $3x + 2 = 4$ in Z_{13}.
18. Find whether 14 is a unit or not in the ring Z_{16}. Establish your claim.
19. Verify whether $f(x) = e^x$ is a permutation on the set of all real numbers or not.
20. Prove that non zero divisors of Z_n are precisely the non zero element that are not relatively prime to n.
21. Show that the cancellation laws hold in a ring R if and only if it has no zero divisors.
22. Show that every field is an integral domain and the converse need not be true in general.
23. Let $GL(n, \mathbb{R})$ be the multiplicative group of all invertible square matrices of order n. Show that $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$ is a homomorphism.
24. Write down all the left cosets of $H = \{\rho_0, \mu_1\}$ in $S_3 = \{\rho_0 = \iota, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ with usual notations for the permutations.
25. Define a binary structure and show that the identity element, if it exists, in a binary structure is unique.
26. Find all the generators of the group $\mathbb{Z}_{32}$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

27. Describe briefly about the dihedral group D_3 and draw its group table.
28. Prove or disprove : Arbitrary intersection of normal subgroups is a subgroup.

29. (i) Show that every subgroup of a cyclic group is a normal subgroup and

    (ii) Give an example of a cyclic group of order n for a given $n > 1$ and one of its non-trivial, proper normal subgroups.

30. Prove that every infinite cyclic group is isomorphic to the group of integers under addition and that if G is cyclic of finite order, then it is isomorphic to $\langle Z_n, +n \rangle$.

31. Show that $|ab| = |ba|$ for all elements a, b in any finite group G.

32. Let R be a ring mid let a be a fixed element of R. Show that $I_a = \{x \in R : ax = 0\}$ is a subring of R.

33. If H is a normal subgroup of G, show that $aHbH = abH$ for all $a, b \in G$.

34. Show that any two fields of quotients of an integral domain are isomorphic.

35. Show that the composition of two group homomorphisms is again a group homomorphism.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

36. (a) If $\phi : G \to G'$ is a group homomorphism, show that $\phi[H] \leq G$ whenever $H \leq G$.

    (b) Show that order of every element in a finite group is finite.

37. (a) Prove Cayley's theorem for groups establishing each claim succinctly.

    (b) Solve $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_3$.

38. (a) Show that $\mathbb{Z}_n$ is a field if and only if n is a prime.

    (b) Establish that the set of all units in a ring with unity form a multiplicative group.

(2 × 13 = 26 marks)